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Large Deflections of Thin Rods under Nonsymmetric Distributed Loads

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Introduction

The analytic approximate solution to nonlinear bending problems of elastic rods under distributed loads has been supplied by Christensen[1] using the Ritz method with one term of a trigonometric series. Such single-term solutions can no longer give a satisfactory approximation when nonsymmetric loads are present. As a remedy, a two-term solution is presented in this note.

Two-Term Galerkin Solution

The governing differential equation for the system shown in Fig. 1 has the form

$$d/ds[EI(d\theta/ds)] - H\sin\theta + V\cos\theta = 0 \qquad (1)$$

where

$$H = H_0 + \int_0^s p(s)\,ds = H_l - \int_s^l p(s)\,ds$$

$$V = V_0 - \int_0^s q(s)\,ds = \int_s^l q(s)\,ds - V_l$$

The approximate solution of Eq. (1) is assumed to be

$$\tilde{\theta} = \theta_1\cos\beta s + \theta_2\cos 2\beta s \qquad \beta = \pi/l \qquad (2)$$

Then, by Galerkin's method,[2] the constants θ_1 and θ_2 are determined by the system of equations

$$\int_0^l NL(\tilde{\theta})\cos\beta s\,ds = 0 \qquad \int_0^l NL(\tilde{\theta})\cos 2\beta s\,ds = 0 \qquad (3)$$

and $NL(\theta)$ represents the nonlinear differential equation (1).

The integration in (3) can be carried out by making use of the trigonometric identities and the expansions of Bessel series.[3] It can be shown that, for a thin rod with a constant flexural rigidity EI under uniformly distributed loads of intensities p_0 and q_0, Eqs. (3), upon integration, lead to the following simultaneous transcendental equations:

$$\left.\begin{aligned} &\theta_1 P_c/Q_0 + (2H_0/Q_0 + m)A_1 + mB_1 + \\ &\qquad (2V_0/Q_0 - 1)C_1 - D_1 = 0 \\ &4\theta_2 P_c/Q_0 + (2H_0/Q_0 + m)A_2 + mB_2 + \\ &\qquad (2V_0/Q_0 - 1)C_2 + D_2 = 0 \\ &\qquad (Q_0 \neq 0) \end{aligned}\right\} \qquad (4)$$

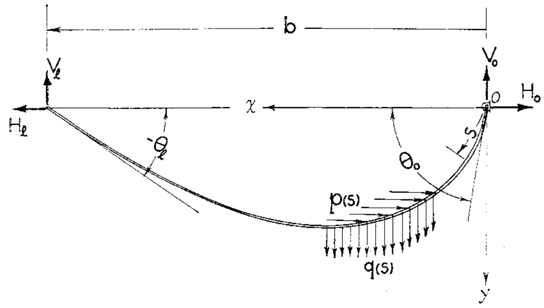

Fig. 1 Simply supported elastic thin rod loaded nonsymmetrically.

where the symbols are defined as follows:

$$P_c = \pi^2 EI/l^2 \qquad Q_0 = q_0 l \qquad m = p_0/q_0$$

$$A_1 = J_1(\theta_1)J_0(\theta_2) + \sum_{k=1}^{\infty}(-1)^k J_{2k}(\theta_2)[J_{4k+1}(\theta_1) - J_{4k-1}(\theta_1)]$$

$$B_1 = \left(\frac{8}{\pi^2}\right)\left\{\sum_{k=1}^{\infty}(-1)^k J_0(\theta_1)J_{2k-1}(\theta_2) \times \frac{(4k-2)^2+1}{[(4k-2)^2-1]^2} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k}J_{2j}(\theta_1)J_{2k-1}(\theta_2) \times \left[\frac{c_1^2+1}{(c_1^2-1)^2} + \frac{c_2^2+1}{(c_2^2-1)^2}\right]\right\}$$

$$c_1 = 4k + 2j - 2 \qquad c_2 = 4k - 2j - 2$$

$$C_1 = \sum_{k=1}^{\infty}(-1)^k J_{2k-1}(\theta_2)[J_{4k-1}(\theta_1) - J_{4k-3}(\theta_1)]$$

$$D_1 = \left(\frac{8}{\pi^2}\right)\left\{\frac{J_0(\theta_2)J_0(\theta_1)}{2} + \sum_{k=1}^{\infty}(-1)^k \times \left[\frac{J_0(\theta_2)J_{2k}(\theta_1)(4k^2+1)}{(4k^2-1)^2} + \frac{J_0(\theta_1)J_{2k}(\theta_2)(16k^2+1)}{(16k^2-1)^2}\right] + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k}J_{2j}(\theta_1)J_{2k}(\theta_2) \times \left[\frac{c_3^2+1}{(c_3^2-1)^2} + \frac{c_4^2+1}{(c_4^2-1)^2}\right]\right\}$$

$$c_3 = 4k + 2j \qquad c_4 = 4k - 2j$$

$$A_2 = J_0(\theta_1)J_1(\theta_2) - \sum_{k=1}^{\infty}(-1)^k J_{2k-1}(\theta_2)[J_{4k-4}(\theta_1) + J_{4k}(\theta_1)]$$

$$B_2 = \left(\frac{8}{\pi^2}\right)\left\{\sum_{k=1}^{\infty}(-1)^k J_0(\theta_2)J_{2k-1}(\theta_1)\left[\frac{(2k-1)^2+4}{[(2k-1)^2-4]^2}\right] + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k} \times J_{2j-1}(\theta_1)J_{2k}(\theta_2) \times \left[\frac{c_5^2+4}{(c_5^2-4)^2} + \frac{c_6^2+4}{(c_6^2-4)^2}\right]\right\}$$

$$c_5 = 4k + 2j - 1 \qquad c_6 = 4k - 2j + 1$$

$$C_2 = J_0(\theta_2)J_2(\theta_1) + \sum_{k=1}^{\infty}(-1)^k J_{2k}(\theta_2)[J_{4k+2}(\theta_1) + J_{4k-2}(\theta_1)]$$

$$D_2 = \frac{8}{\pi^2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{j+k}J_{2j-1}(\theta_1)J_{2k-1}(\theta_2) \times \left[\frac{c_7^2+4}{(c_7^2-4)^2} + \frac{c_8^2+4}{(c_8^2-4)^2}\right]$$

$$c_7 = 4k + 2j - 3 \qquad c_8 = 4k - 2j - 1$$

In Eqs. (4), the force V_0 is not an independent quantity. It is related to the external loads by the condition of equilibrium. Summing moments about the left end point yields

$$V_0 b - \int_0^l q_0(b - x)ds + \int_0^l p_0 y\,ds = V_0 b - q_0 bl + q_0\int_0^l\left[\int_0^s \cos\tilde{\theta}\,ds\right]ds + p_0\int_0^l\left[\int_0^s \sin\tilde{\theta}\,ds\right]ds = 0 \qquad (5)$$

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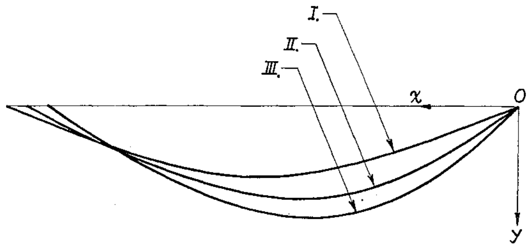

CASE	P_c/Q_o	H_o/Q_o	m	θ_1	θ_2
I.	0.244	0.796	0	22°	0°
II.	0.244	0.069	0.727	32°	4°
III.	0.226	-0.228	1.035	40°	6°

Fig. 2 Symmetric and nonsymmetric large deformation of rod with fixed length.

where b is the distance between the two ends of the rod, and its magnitude is

$$b = \int_0^l \cos\tilde{\theta}\, ds \cong lJ_0(\theta_1)J_0(\theta_2) \qquad (6)$$

Performing the indicated integration in Eq. (5) with the use of Bessel series, the expression of V_0 may be put in the form

$$V_0 = q_0 l - (q_0 l^2/b)G_1 - (p_0 l^2/b)G_2 \qquad (7)$$

where

$$G_1 = J_0(\theta_1)J_0(\theta_2) + \sum_{k=1}^{\infty} (-1)^k J_{4k}(\theta_1)J_{2k}(\theta_2) - \frac{4}{\pi^2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (-1)^{j+k} \times J_{2j-1}(\theta_1)J_{2k-1}(\theta_2) \times \left[\frac{1}{(4k+2j-3)^2} + \frac{1}{(4k-2j-1)^2}\right]$$

$$G_2 = \frac{4}{\pi^2}\left\{\sum_{k=1}^{\infty} (-1)^k \left[\frac{\pi^2}{4} J_{4k-2}(\theta_1)J_{2k-1}(\theta_2) - \frac{J_0(\theta_2)J_{2k-1}(\theta_1)}{(2k-1)^2}\right] - \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} (-1)^{j+k} J_{2j-1}(\theta_1)J_{2k}(\theta_2) \times \left[\frac{1}{(4k+2j-1)^2} + \frac{1}{(4k-2j+1)^2}\right]\right\}$$

The infinite series in Eqs. (4) and (7) converges rapidly, and only a few terms will suffice to give a satisfactory accuracy. Eliminating V_0 from Eqs. (4) by the substitution of (7) and discarding the terms having negligible contribution will yield a set of simplified simultaneous equations which can be solved either graphically or by iteration procedures.[4] The solution can be made much easier by finding the approximate value of θ_1 in advance from the following single-term condition:

$$\frac{\theta_1 P_c}{Q_0} + \left(\frac{2H_0}{Q_0} + m\right) J_1(\theta_1) - \frac{8}{\pi^2}\left\{\frac{J_0(\theta_1)}{2} + \sum_{k=1}^{\infty} (-1)^k J_{2k}(\theta_1) \times \frac{4k^2+1}{(4k^2-1)^2}\right\} = 0 \qquad (8)$$

Equation (8) is obtained by carrying out the Galerkin's solution with only the first term of Eq. (2). The deformed shapes found by solving Eqs. (4) with the aid of (8) for a rod of fixed length are shown in Fig. 2.

Concluding Remarks

The two-term solution presented previously has some practical importance. One of the difficult problems encountered in industry is to determine the relationship between the internal stresses and the boundary forces of a heavy suspended elastic rod whose two supports are not on the same level. With the methods given in this note, the problem can now be treated analytically by choosing a set of inclined coordinates. Further generalization can readily be made to account for the effects of variable rigidity and to include the cases where the distributed loads are functions of s.

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An Approximate Solution to Hypersonic Blunt-Body Problem

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Nomenclature

H = entropy gradient parameter
M = Mach number
p = $\bar{p}/\bar{\rho}_\infty \bar{U}_\infty^2$, nondimensional pressure
P_x = $\frac{1}{\sin^2\theta}\int_0^\theta p_b(\xi)\sin\xi\cos\xi\, d\xi$
R = $\bar{R}/\bar{R}_b$, dimensionless radial distance
u = $\bar{u}/\bar{U}_\infty$, dimensionless tangential velocity component
$\bar{U}_\infty$ = freestream velocity
v = $\bar{v}/\bar{U}_\infty$, dimensionless radial velocity component
α = shock inclination from direction normal to freestream
γ = ratio of specific heats
Δ = shock-layer thickness
η, κ = pressure distribution parameters
θ = angle between R and axis of symmetry
ξ = dummy integration variable
ρ = $\bar{\rho}/\bar{\rho}_\infty$, dimensionless density
ω = flow deflection angle behind shock wave

Subscripts

b = quantity evaluated at body surface
s = quantity evaluated immediately behind shock wave
1 = quantity evaluated on stagnation streamline
∞ = freestream quantity

Superscripts

* = quantity evaluated on ray through sonic point on body
— = physical quantity

I. Introduction

The approximate method outlined herein for solution of the direct hypersonic blunt body problem does not use a step-by-step advance of the solution and thus avoids the problems of error accumulation and of singular point instabilities inherent in such methods as that formulated by Belotserkovskii.[1] Integral equations are formulated ex-

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